# ESPRIT algorithm and DOA estimation

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# Outline

# Outcomes

- Learn Harmonic modeling of signals
- Get an overview of Frequency estimation techniques using Harmonic Model
- Know about theoretical and practical correlation matrices
- Understand the details, working and flow of ESPRIT algorithm
- Understand it's application to DOA estimation in array processing

# Overview Frequency Estimation techniques

## The problem:

Given a snapshot of samples $\{x(1), ...., x(N)\}$,
how to estimate the frequencies present in the data(signal).

# Harmonic model

- In many applications signals of interest are complex exponentials contained in noise.
- e.g. formant frequencies in speech processing, moving targets in radar, spatially propagating signals in array processing.

# Harmonic model

- Signal containing P complex exponentials in noise:

$$x(n) = \sum_{p=1}^{P} (\alpha_p e^{j2\pi n f_p}) + w(n)$$

- Discrete time frequency of $p^{th}$ component is

$$f_p = \frac{\omega_p}{2\pi} = \frac{F_p}{F_s}$$

phase component of each complex exponential is contained in the amplitude, i.e.

$$\alpha_p = |\alpha_p| e^{j\psi_p}$$

- The power spectrum of complex exponentials is commonly referred to as line spectrum

# Autocorrelation matrix of harmonic model

- Consider a M length snapshot of the signal i.e

$$\mathbf{x}(n) = [x(n)\ x(n+1)\ x(n+2)\ \ldots\ x(n+M-1)\ ]^T$$

- vector representation of above snapshot

$$\mathbf{x}(n) = \sum_{p=1}^{P} \alpha_p \mathbf{v}(f_p) e^{j2\pi n f_p} + \mathbf{w}(n) = \mathbf{s}(n) + \mathbf{w}(n)$$

where; $\mathbf{v}(f) = [1\ e^{2\pi f}\ \ldots\ e^{2\pi(M-1)f}]^T$

# Autocorrelation matrix of harmonic model

- The autocorrelation matrix of this model

$$\mathbf{R}_x = E\{x(n)x^H(n)\} = \mathbf{R}_s + \mathbf{R}_w$$

- 

$$\mathbf{R}_x = \sum_{p=1}^{P} |\alpha_p|^2 \mathbf{v}(f_p)\mathbf{v}^H(f_p) + \sigma_w^2 \mathbf{I} = \mathbf{VAV}^H + \sigma_w^2 \mathbf{I}$$

where

$$\mathbf{V} = [\mathbf{v}(f_1)\ \mathbf{v}(f_2)\ \ldots\ \mathbf{v}(f_P)\ ]$$

$$\mathbf{A} = \begin{bmatrix} |\alpha_1|^2 & 0 & \ldots & 0 \\ 0 & |\alpha_2|^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & |\alpha_P|^2 \end{bmatrix}$$

# Signal subspace and noise subspace

- $\mathbf{R}_x = E\{\mathbf{x}(n)\mathbf{x}^H(n)\} = \mathbf{R}_s + \mathbf{R}_w$
  $\mathbf{R}_s$ is rank-defficient for,
  P(# complex exponentials) $\leq$ M(time window length).
- The autocorrelation matrix can also be written in terms of eigen decomposition

$$\mathbf{R}_x = \sum_{m=1}^{M} \lambda_m \mathbf{q}_m \mathbf{q}_m^H = \mathbf{Q}\Lambda\mathbf{Q}^H$$

  $\lambda_m$ are eigen values in descending order $\mathbf{q}_m$ are corresponding eigen vectors
- $\lambda_m = M|\alpha_m|^2 + \sigma_w^2$ ; for m$\leq$P
  $\lambda_m = \sigma_w^2$ ; for m>P
- $\Rightarrow \mathbf{R}_x = \mathbf{Q}_s\Lambda_s\mathbf{Q}_s^H + \sigma_w^2\mathbf{Q}_w\mathbf{Q}_w^H$

# Signal subspace and noise subspace

- $\Rightarrow \mathbf{R}_x = \mathbf{Q}_s \Lambda_s \mathbf{Q}_s^H + \sigma_w^2 \mathbf{Q}_w \mathbf{Q}_w^H$
- Subspace spanned by $\mathbf{Q}_s$ columns $\leftarrow$ Signal Subspace
  Subspace spanned by $\mathbf{Q}_w$ columns $\leftarrow$ Noise Subspace
- The above two sub spaces are orthogonal to each other, since the correlation matrix is hermitian symmetric
- Time window frequency vectors $\mathbf{v}(f_p)$'s must lie completely in the signal subspace
- Basis for subspace based frequency estimation methods

# Theoretical and Practical correlation matrix

- Theoretical Correlation matrix:

$$\mathbf{R}_x = E\{\mathbf{x}(n)\mathbf{x}^H(n)\}$$

- Practical Correlation matrix:

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}^T(0) \\ \mathbf{x}^T(1) \\ \vdots \\ \mathbf{x}^T(N-1) \end{bmatrix} = \begin{bmatrix} x(0) & x(1) & \dots & x(M-1) \\ x(1) & x(2) & \dots & x(M) \\ \vdots & \vdots & \vdots & \vdots \\ x(N-1) & x(N) & \dots & x(N+M-2) \end{bmatrix}$$

- $\Rightarrow \mathbf{R}_x = \dfrac{\mathbf{X}^H\mathbf{X}}{N}$

# Preliminaries

## ESPRIT Algorithm:(Estimation of signal parameters via rotational invariance techniques)

- Differs from other subspace methods, in the sense that subspace is estimated from data matrix($\mathbf{X}$), rather than correlation matrix($\mathbf{R}_x$).
- The essence of ESPRIT lies in the **rotational property** between staggered sub spaces.

# Preliminaries

- Harmonic model:

$$\mathbf{x}(n) = \sum_{p=1}^{P} \alpha_p \mathbf{v}(f_p) e^{j2\pi n f_p} + \mathbf{w}(n) = \boldsymbol{V}\Phi^n \boldsymbol{\alpha} + \mathbf{w}(n) = \mathbf{s}(n) + \mathbf{w}(n)$$

where, $\Phi = diag\{\Phi_1,\ \Phi_2,\ \ldots,\ \Phi_P\} =$

$$\begin{bmatrix} e^{j2\pi f_1} & 0 & \ldots & 0 \\ 0 & e^{j2\pi f_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & e^{j2\pi f_P} \end{bmatrix}$$

- frequencies of the complex exponentially describe this rotation matrix, the frequency estimates can be obtained by finding $\Phi$

# Preliminaries

- Consider two overlapping subwindows of length $M - 1$ within the length $M$ time window vector

$$\mathbf{s}(n) = \begin{bmatrix} \mathbf{s}_{M-1}(n) \\ s(n+M-1) \end{bmatrix} = \begin{bmatrix} s(n) \\ \mathbf{s}_{M-1}(n+1) \end{bmatrix}$$

where; $\mathbf{s}_{M-1}(n) = M - 1$ *length subwindow of* $\mathbf{s}(n) = \mathbf{V}_{M-1}\Phi^n\alpha$

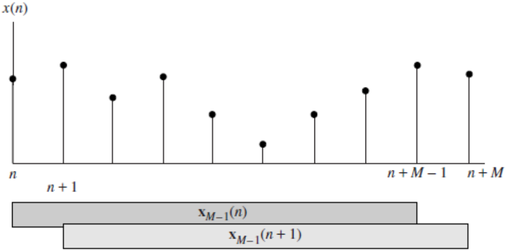



Figure: Time-staggered, overlapping windows used by the ESPRIT algorithm

# Preliminaries

- Consider the above staggered windows, i.e.

$$\mathbf{s}(n) = \begin{bmatrix} \mathbf{s}_{M-1}(n) \\ s(n+M-1) \end{bmatrix} = \begin{bmatrix} s(n) \\ \mathbf{s}_{M-1}(n+1) \end{bmatrix}$$

  where; $\mathbf{s}_{M-1}(n) = M-1$ *length subwindow of* $\mathbf{s}(n) = \mathbf{V}_{M-1}\Phi^n\boldsymbol{\alpha}$

- We define the matrices:

$$\mathbf{V}_1 = \mathbf{V}_{M-1}\Phi^n$$

  and

$$\mathbf{V}_2 = \mathbf{V}_{M-1}\Phi^{n+1}$$

  where $\mathbf{V}_1$ and $\mathbf{V}_2$ correspond to staggered windows

- From the above matrices, we can see that

$$\mathbf{V}_2 = \mathbf{V}_1\Phi$$

  Each of this two matrices spans a different, though related, $M-1$-dim subspace

# Least squares ESPRIT

- Now, suppose that we have a data matrix $\mathbf{X}$ , with N snapshots of length M time window.
  $\Rightarrow$ Using singular value decomposition:

$$\mathbf{X} = \mathbf{L}\Sigma\mathbf{U}^H$$

  where,
  $\mathbf{L}$ is an $N \times N$ matrix of left singular vectors
  $\mathbf{U}$ is $M \times M$ matrix of right singular vectors
  $\Sigma$ is $N \times M$ singular values along the diagonal

# Least squares ESPRIT

- Using singular value decomposition:

$$\mathbf{X} = \mathbf{L}\Sigma\mathbf{U}^H$$

- The squared magnitudes of singular values are equal to the eigen values of correlation matrix($\mathbf{R}_x$) scaled by a factor N, and colummns of $\mathbf{U}$ are corresponding eigen vectors.
- Thus, $\mathbf{U}$ forms an orthonormal basis for underlying M dimensional space
- This sub space can be partitioned into signal and noise subspace

$$\mathbf{U} = [\mathbf{U}_s | \mathbf{U}_n]$$

# Least squares ESPRIT

- This sub space can be partitioned into signal and noise subspace
$$\mathbf{U} = [\mathbf{U}_s | \mathbf{U}_n]$$
- $\mathbf{U}_s$ is matrix of right handed singular vectors corresponding to singular values of P largest magnitudes.
- The matrices $\mathbf{V}$ and $\mathbf{U}_s$ span the same subspace, $\Rightarrow$ There exists a invertible transformation $\mathbf{T}$, that maps $\mathbf{U}_s$ to $\mathbf{V}$ i.e.
$$\mathbf{V} = \mathbf{U}_s \mathbf{T}$$
- Similar staggering matrices of $\mathbf{V}$ can be defined for $\mathbf{U}_s$, say, $\mathbf{U}_1$ and $\mathbf{U}_2$

# Least squares ESPRIT

- staggered matrices of $\mathbf{U}_s$: $\mathbf{U}_1$ and $\mathbf{U}_2$
  staggered matrices of $\mathbf{V}$: $\mathbf{V}_1$ and $\mathbf{V}_2$
  holds the relation,

$$\mathbf{V}_1 = \mathbf{U}_1\mathbf{T} \textit{ and } \mathbf{V}_2 = \mathbf{U}_2\mathbf{T}$$

- Similar to $\mathbf{V}_1$ and $\mathbf{V}_2$, $\mathbf{U}_1$ and $\mathbf{U}_2$ are related by a rotation matrix $\Psi$

$$\mathbf{U}_2 = \mathbf{U}_1\Psi$$

- Now, we solve for $\Psi$ using least squares,

$$\Psi = (\mathbf{U}_1^H\mathbf{U}_1)^{-1}\mathbf{U}_1^H\mathbf{U}_2$$

# Least squares ESPRIT

- $$\mathbf{V}_2 = \mathbf{U}_2\mathbf{T} = \mathbf{U}_1\Psi\mathbf{T}$$
  $$\mathbf{V}_2 = \mathbf{V}_1\Phi = \mathbf{U}_1\mathbf{T}\Phi$$
- From above two equations, we can write
  $$\Psi\mathbf{T} = \mathbf{T}\Phi$$
  $$\Rightarrow \Psi = \mathbf{T}\Phi\mathbf{T}^{-1}$$
  this equation can be recognized as relationship between eigen vectors and values of $\Psi$
- therefore, the diagonal elements of $\Phi$ are simply the eigen values of $\Psi$
- as a result the estimates of frequency are
  $$\hat{f}_p = \frac{\phi_p}{2\pi}$$

# Total least squares ESPRIT

- form a matrix made up of staggered signal subspace matrices $\mathbf{U}_1$ and $\mathbf{U}_2$, placed side by side, and perform SVD i.e.

$$[\mathbf{U}_1 \ \mathbf{U}_2] = \widetilde{\mathbf{L}}\widetilde{\Sigma}\widetilde{\mathbf{U}}^H$$

  we then operate on $2P \times 2P$ matrix $\widetilde{\mathbf{U}}$

- $$\widetilde{\mathbf{U}} = \begin{bmatrix} \widetilde{\mathbf{U}}_{11} & \widetilde{\mathbf{U}}_{12} \\ \widetilde{\mathbf{U}}_{21} & \widetilde{\mathbf{U}}_{22} \end{bmatrix}$$

- The TLS solution for the subspace rotation matrix $\Psi$ is

$$\Psi_{tls} = -\widetilde{\mathbf{U}}_{12}\widetilde{\mathbf{U}}_{22}^{-1}$$

# Complete flow of ESPRIT algorithm

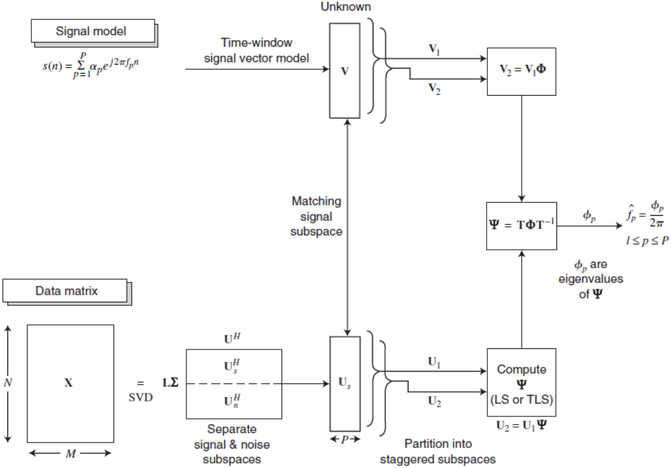
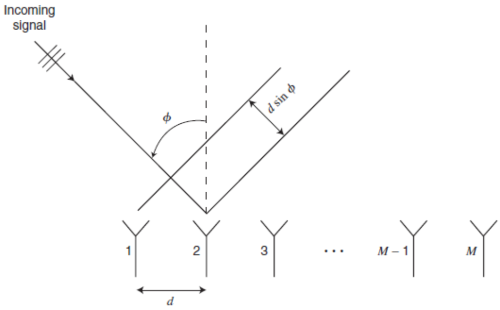



Figure: Flow of ESPRIT algorithm

# DOA and Array processing




- Estimate source location using sensor arrays
- The delay in the signal received by $m^{th}$ element of ULA, compared to first element, from a $p^{th}$ source is

$$\Delta_{p,m} = \frac{(m-1)\, d\, sin(\phi_p)}{c}$$

# DOA estimation

- The delay in the signal received by $m^{th}$ element of ULA, compared to first element, from a $p^{th}$ source is

$$\Delta_{p,m} = \frac{(m-1)\, d\, sin(\phi_p)}{c}$$

- The signal received by the whole ULA of length M, from a $p^{th}$ source is

$$\mathbf{a}(\Delta_p)\, s_p(t) = \begin{pmatrix} 1 \\ e^{j\omega_c\, \Delta_{p,1}} \\ e^{j\omega_c\, \Delta_{p,2}} \\ e^{j\omega_c\, \Delta_{p,3}} \\ \vdots \\ e^{j\omega_c\, \Delta_{p,M}} \end{pmatrix} s_p(t)$$

# DOA estimation

- For all the P sources and M length ULA, the signal model is a superposition of signals from all the P sources
- The received signal at ULA is represented as:

$$\mathbf{y}(n) = [\mathbf{a}(\Delta_1), \mathbf{a}(\Delta_2), \ldots, \mathbf{a}(\Delta_P)] \begin{bmatrix} s_1(n) \\ s_2(n) \\ s_3(n) \\ \vdots \\ s_P(n) \end{bmatrix} + \mathbf{w}(n)$$

- This is similar to the harmonic model, that we had considered for applying ESPRIT algorithm
- i.e. $\mathbf{V} = [\mathbf{a}(\Delta_1), \mathbf{a}(\Delta_2), \ldots, \mathbf{a}(\Delta_P)]$
- This $\mathbf{V}$ can be split into staggered sub-spaces $\mathbf{V}_1$ *and* $\mathbf{V}_2$

# Analogy: "ESPRIT" and "ESPRIT for DOA"

$$\mathbf{s}(n) = \begin{bmatrix} \sum_{p=1}^{P} \alpha_p e^{j2\pi f_p n}.1 \\ \sum_{p=1}^{P} \alpha_p e^{j2\pi f_p n}.e^{j2\pi f_p} \\ \dots \\ \sum_{p=1}^{P} \alpha_p e^{j2\pi f_p n}.e^{j2\pi (M-1) f_p} \end{bmatrix} = \mathbf{V}\Phi^n \alpha$$

$$\mathbf{z}(n) = \begin{bmatrix} \sum_{p=1}^{P} s_p(n).1 \\ \sum_{p=1}^{P} s_p(n).e^{j\Delta_p} \\ \dots \\ \sum_{p=1}^{P} s_p(n).e^{j(M-1)\Delta_p} \end{bmatrix} = \mathbf{V}\mathbf{s}(n)$$

# Summary

- The ESPRIT algorithm is a parametric spectrum estimation method, it is one of the super-resolution method.
- it turns out that we can use this method for DOA estimation, since both problem are one and the same, just an interchange play between frequency and phase delay terms
- ESPRIT makes use of rotational property of staggered sub spaces and comparing a match between the harmonic model and SVD of data matrix.
- It involves SVD and least squares concept.

# Thank you !